Isomorphism among the 95 Families of Weighted K3 hypersurfaces

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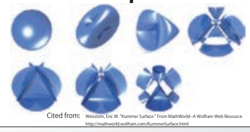

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Joint work with Prof. M. Kobayashi

1. Introduction

Some families of weighted $K3$ hypersurfaces have the isometric Picard lattices. (see Table 1)

A natural question arises :
Are families with the isometric Picard lattices isomorphic?

The Picard lattices do not determine isomorphism between $K3$ surfaces, but we have an idea. (see Figure 1)

Famous 95 families of weighted $K3$ hypersurfaces

1. $\boldsymbol{P}(1,1,1,1) \supset (4)$	25. $\boldsymbol{P}(1,1,3,4) \supset (9)$	49. $\boldsymbol{P}(2,5,14,21) \supset (42)$	73. $\boldsymbol{P}(7,8,10,25) \supset (50)$
2. $\boldsymbol{P}(2,3,3,4) \supset (12)$	26. $\boldsymbol{P}(2,4,5,9) \supset (20)$	50. $\boldsymbol{P}(1,4,10,15) \supset (30)$	74. $\boldsymbol{P}(4,5,7,16) \supset (32)$
3. $\boldsymbol{P}(1,1,2,2) \supset (6)$	27. $\boldsymbol{P}(2,3,8,11) \supset (24)$	51. $\boldsymbol{P}(1,5,12,18) \supset (36)$	75. $\boldsymbol{P}(2,4,5,11) \supset (22)$
4. $\boldsymbol{P}(1,3,4,4) \supset (12)$	28. $\boldsymbol{P}(1,3,7,10) \supset (21)$	52. $\boldsymbol{P}(7,8,9,12) \supset (36)$	76. $\boldsymbol{P}(2,5,6,13) \supset (26)$
5. $\boldsymbol{P}(1,1,1,3) \supset (6)$	29. $\boldsymbol{P}(4,5,6,15) \supset (30)$	53. $\boldsymbol{P}(3,4,5,6) \supset (18)$	77. $\boldsymbol{P}(1,5,7,13) \supset (26)$
6. $\boldsymbol{P}(1,2,2,5) \supset (10)$	30. $\boldsymbol{P}(5,7,8,20) \supset (40)$	54. $\boldsymbol{P}(3,5,6,7) \supset (21)$	78. $\boldsymbol{P}(1,4,6,11) \supset (22)$
7. $\boldsymbol{P}(1,1,2,4) \supset (8)$	31. $\boldsymbol{P}(3,4,5,12) \supset (24)$	55. $\boldsymbol{P}(2,5,6,7) \supset (20)$	79. $\boldsymbol{P}(2,5,9,16) \supset (32)$
8. $\boldsymbol{P}(1,2,3,6) \supset (12)$	32. $\boldsymbol{P}(2,2,3,7) \supset (14)$	56. $\boldsymbol{P}(5,6,8,11) \supset (30)$	80. $\boldsymbol{P}(4,5,13,22) \supset (44)$
9. $\boldsymbol{P}(1,4,5,10) \supset (20)$	33. $\boldsymbol{P}(2,3,4,9) \supset (18)$	57. $\boldsymbol{P}(4,5,6,9) \supset (24)$	81. $\boldsymbol{P}(2,3,8,13) \supset (26)$
10. $\boldsymbol{P}(1,1,4,6) \supset (12)$	34. $\boldsymbol{P}(2,6,7,15) \supset (30)$	58. $\boldsymbol{P}(1,4,5,6) \supset (16)$	82. $\boldsymbol{P}(1,3,7,11) \supset (22)$
11. $\boldsymbol{P}(2,3,10,15) \supset (30)$	35. $\boldsymbol{P}(3,4,7,14) \supset (28)$	59. $\boldsymbol{P}(1,5,7,8) \supset (21)$	83. $\boldsymbol{P}(4,5,18,27) \supset (54)$
12. $\boldsymbol{P}(1,2,6,9) \supset (18)$	36. $\boldsymbol{P}(2,3,5,10) \supset (20)$	60. $\boldsymbol{P}(1,4,6,7) \supset (18)$	84. $\boldsymbol{P}(5,6,7,9) \supset (27)$
13. $\boldsymbol{P}(1,3,8,12) \supset (24)$	37. $\boldsymbol{P}(1,3,4,8) \supset (16)$	61. $\boldsymbol{P}(4,6,7,11) \supset (28)$	85. $\boldsymbol{P}(2,3,4,5) \supset (14)$
14. $\boldsymbol{P}(1,6,14,21) \supset (42)$	38. $\boldsymbol{P}(1,6,8,15) \supset (30)$	62. $\boldsymbol{P}(3,4,5,8) \supset (20)$	86. $\boldsymbol{P}(4,5,7,9) \supset (25)$
15. $\boldsymbol{P}(3,3,4,5) \supset (15)$	39. $\boldsymbol{P}(1,3,5,9) \supset (18)$	63. $\boldsymbol{P}(1,2,3,4) \supset (10)$	87. $\boldsymbol{P}(1,3,4,5) \supset (13)$
16. $\boldsymbol{P}(3,6,7,8) \supset (24)$	40. $\boldsymbol{P}(1,2,4,7) \supset (14)$	64. $\boldsymbol{P}(3,4,7,10) \supset (24)$	88. $\boldsymbol{P}(2,5,9,11) \supset (27)$
17. $\boldsymbol{P}(2,3,5,5) \supset (15)$	41. $\boldsymbol{P}(2,3,7,12) \supset (24)$	65. $\boldsymbol{P}(3,5,11,14) \supset (33)$	89. $\boldsymbol{P}(1,2,3,5) \supset (11)$
18. $\boldsymbol{P}(1,2,3,3) \supset (9)$	42. $\boldsymbol{P}(1,1,3,5) \supset (10)$	66. $\boldsymbol{P}(1,1,2,3) \supset (7)$	90. $\boldsymbol{P}(4,6,7,17) \supset (34)$
19. $\boldsymbol{P}(1,2,2,3) \supset (8)$	43. $\boldsymbol{P}(3,4,11,18) \supset (36)$	67. $\boldsymbol{P}(2,3,7,9) \supset (21)$	91. $\boldsymbol{P}(5,6,8,19) \supset (38)$
20. $\boldsymbol{P}(1,6,8,9) \supset (24)$	44. $\boldsymbol{P}(1,2,5,8) \supset (16)$	68. $\boldsymbol{P}(3,4,10,13) \supset (30)$	92. $\boldsymbol{P}(3,5,11,19) \supset (38)$
21. $\boldsymbol{P}(1,1,1,2) \supset (5)$	45. $\boldsymbol{P}(1,4,9,14) \supset (28)$	69. $\boldsymbol{P}(2,3,4,7) \supset (16)$	93. $\boldsymbol{P}(3,4,10,17) \supset (34)$
22. $\boldsymbol{P}(1,3,5,6) \supset (15)$	46. $\boldsymbol{P}(5,6,22,33) \supset (66)$	70. $\boldsymbol{P}(2,3,5,8) \supset (18)$	94. $\boldsymbol{P}(3,4,5,7) \supset (19)$
23. $\boldsymbol{P}(2,2,3,5) \supset (12)$	47. $\boldsymbol{P}(3,4,14,21) \supset (42)$	71. $\boldsymbol{P}(1,3,4,7) \supset (15)$	95. $\boldsymbol{P}(2,3,5,7) \supset (17)$
24. $\boldsymbol{P}(1,2,4,5) \supset (12)$	48. $\boldsymbol{P}(3,5,16,24) \supset (48)$	72. $\boldsymbol{P}(1,2,5,7) \supset (15)$	

Table 1 : The same-coloured families have the isometric Picard lattices.

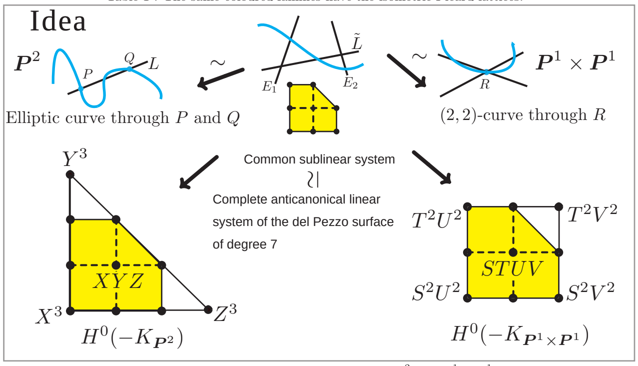

Figure 1 : There exist anticanonical sublinear systems of $\boldsymbol{P}^2$ and $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ which are isomorphic to the complete anticanonical linear system of a del Pezzo surface of degree 7.

2. Set-ups

The *Picard lattice* is the Picard group of a $K3$ surface with a cup product.

Definition. Δ : 3-dimensional integral convex polytope with $0 \in \mathrm{Int}\Delta$,

$$\Delta^\circ := \left\{ y \in (\boldsymbol{R}^3)^* | \langle x, y\rangle \geq -1, \forall x \in \Delta \right\} \quad \text{the } \textit{polar dual of } \Delta,$$

$$\Delta \text{ is } \textit{reflexive.} \overset{\text{def.}}{\iff} \Delta^\circ \text{ is integral.}$$

Notations. $a = (a_0, a_1, a_2, a_3)$: well-posed weight,
Δ : 3-dimensional integral convex polytope,
1) $\boldsymbol{P}(a)$: the weighted projective space,
2) $M(a) := \left\{ (m_0, m_1, m_2, m_3) \in \boldsymbol{Z}^4 \,\middle|\, \sum_{i=0}^3 a_i m_i = 0 \right\}$,
3) $\Delta(a) := \{(m_0, m_1, m_2, m_3) \in M(a) \otimes \boldsymbol{R} \mid m_i \geq -1\}$.
4) $\boldsymbol{P}(\Delta)$: projective toric variety obtained by Δ,
5) Δ : reflexive $\rightsquigarrow$ $\Lambda(\Delta)$ Picard lattice of $\boldsymbol{P}(\Delta)$,
a : weight in 95 families $\rightsquigarrow$ $\Lambda(a)$ Picard lattice of $\boldsymbol{P}(a)$.

Fact.(V.V.Batyrev) Δ : 3-dimensional integral convex polytope,

Δ is reflexive. $\Leftrightarrow$ There exists an irreducible anticanonical divisor of $\boldsymbol{P}(\Delta)$with at worst ADE singularities.

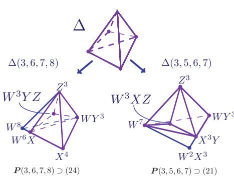

Figure 2

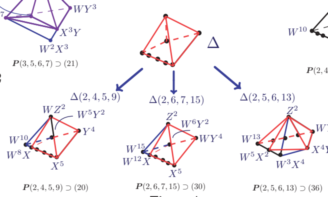

Figure 4

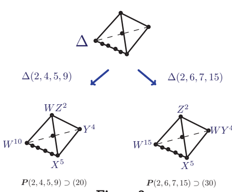

Figure 3

3. Main Result

Take weights a and b from the 95 families with $\Lambda(a) \simeq \Lambda(b)$.

Then, there exist anticanonical sublinear systems D_a and D_b of $\boldsymbol{P}(a)$ and $\boldsymbol{P}(b)$, respectively, and an isomorphism $\phi : D_a \to D_b$ satisfying the followings:
(1) If $X \in D_a$ is a $K3$ surface, then $\phi(X) \in D_b$ is also $K3$, and vice versa,
(2) The Picard lattices of these $K3$ surfaces are isometric.

In other words, there exist a reflexive polytope $\Delta(\subset \Delta(a), \Delta(b))$ and a group isomorphism $M(a) \to M(b)$ such that
(1) Associated birational maps $\boldsymbol{P}(a) \dashrightarrow \boldsymbol{P}(\Delta)$ and $\boldsymbol{P}(b) \dashrightarrow \boldsymbol{P}(\Delta)$ map generic anticanonical members of $\boldsymbol{P}(a)$ and $\boldsymbol{P}(b)$ to those of $\boldsymbol{P}(\Delta)$,
(2) $\Lambda(a) \simeq \Lambda(b) \simeq \Lambda(\Delta)$.

A polytope Δ and associated birational maps can be taken explicitely. (see Table 2 and 3, and Figure 2, 3 and 4).

No.	Families	The vertices of Δ	Picard lattice
13	$\boldsymbol{P}(1,3,8,12) \supset (24)$	$Z^2, W^{24}, W^3X^7, WX^5Y, Y^3, X^4Z$	$E_6 \perp U$
72	$\boldsymbol{P}(1,2,5,7) \supset (15)$	$WZ^2, W^{15}, WX^7, X^5Y, Y^3, X^4Z$	(8)
50	$\boldsymbol{P}(1,4,10,15) \supset (30)$	$Z^2, W^{30}, W^2X^7, X^5Y, Y^3$	$E_7 \perp U$
82	$\boldsymbol{P}(1,3,7,11) \supset (22)$	$Z^2, W^{22}, WX^7, X^5Y, WY^3$	(9)
9	$\boldsymbol{P}(1,4,5,10) \supset (20)$	$W^{20}, X^5, Z^2, Y^2Z, WXY^3, W^5Y^3$	$T_{2,5,5}$
71	$\boldsymbol{P}(1,3,4,7) \supset (15)$	$W^{15}, X^5, WZ^2, Y^2Z, XY^3, W^3Y^3$	(10)
14	$\boldsymbol{P}(1,6,14,21) \supset (42)$	Z^2, Y^3, X^7, W^{42}	$E_8 \perp U$
28	$\boldsymbol{P}(1,3,7,10) \supset (21)$	WZ^2, Y^3, X^7, W^{21}	(10)
45	$\boldsymbol{P}(1,4,9,14) \supset (28)$	Z^2, WY^3, X^7, W^{28}	
51	$\boldsymbol{P}(1,5,12,18) \supset (36)$	Z^2, Y^3, WX^7, W^{36}	
38	$\boldsymbol{P}(1,6,8,15) \supset (30)$	$Z^2, W^{30}, X^5, XY^3, W^6Y^3$	$E_8 \perp A_1 \perp U$
77	$\boldsymbol{P}(1,5,7,13) \supset (26)$	$Z^2, W^{26}, WX^5, XY^3, W^5Y^3$	(11)
20	$\boldsymbol{P}(1,6,8,9) \supset (24)$	$W^6Z^2, W^{24}, X^4, XZ^2, Y^3$	$E_8 \perp A_2 \perp U$
59	$\boldsymbol{P}(1,5,7,8) \supset (21)$	$W^5Z^2, W^{21}, WX^4, XZ^2, Y^3$	(12)
26	$\boldsymbol{P}(2,4,5,9) \supset (20)$	WZ^2, W^{10}, X^5, Y^4	$D_8 \perp D_4 \perp U$
34	$\boldsymbol{P}(2,6,7,15) \supset (30)$	Z^2, W^{15}, X^5, WY^4	(14)
26	$\boldsymbol{P}(2,4,5,9) \supset (20)$	$WZ^2, W^5Y^2, Y^4, X^5, W^8X$	$D_8 \perp D_4 \perp U$
34	$\boldsymbol{P}(2,6,7,15) \supset (30)$	$Z^2, W^8Y^2, WY^4, X^5, W^{12}X$	(14)
76	$\boldsymbol{P}(2,5,6,13) \supset (26)$	$Z^2, W^8X^2, X^4Y, WY^4, W^{13}$	
27	$\boldsymbol{P}(2,3,8,11) \supset (24)$	WZ^2, W^{12}, X^8, Y^3	$E_8 \perp D_4 \perp U$
49	$\boldsymbol{P}(2,5,14,21) \supset (42)$	Z^2, W^{21}, WX^8, Y^3	(14)
16	$\boldsymbol{P}(3,6,7,8) \supset (24)$	$Z^3, W^3YZ, W^6X, X^4, WY^3$	$E_8 \perp (A_2)^3 \perp U$
54	$\boldsymbol{P}(3,5,6,7) \supset (21)$	$Z^3, W^3XZ, W^7, WY^3, X^3Y$	(16)
43	$\boldsymbol{P}(3,4,11,18) \supset (36)$	Z^2, W^{12}, X^9, WY^3	$E_8 \perp E_6 \perp U$
48	$\boldsymbol{P}(3,5,16,24) \supset (48)$	Z^2, W^{16}, WX^9, Y^3	(16)
43	$\boldsymbol{P}(3,4,11,18) \supset (36)$	$Z^2, W^6Z, W^8X^3, X^9, WY^3, W^7XY$	$E_8 \perp E_6 \perp U$
48	$\boldsymbol{P}(3,5,16,24) \supset (48)$	$Z^2, W^8Z, W^{11}X^3, WX^9, Y^3, W^9XY$	(16)
88	$\boldsymbol{P}(2,5,9,11) \supset (27)$	$XZ^2, W^8Z, W^{11}X, WX^5, Y^3, W^9Y$	
68	$\boldsymbol{P}(3,4,10,13) \supset (30)$	$XZ^2, X^5Y, W^2X^6, Y^3, W^{10}$	$E_8 \perp E_7 \perp U$
83	$\boldsymbol{P}(4,5,18,27) \supset (54)$	$Z^2, W^9Y, W^{11}X^2, Y^3, WX^{10}$	(17)
92	$\boldsymbol{P}(3,5,11,19) \supset (38)$	$Z^2, W^9Y, W^{11}X, XY^3, WX^7$	
30	$\boldsymbol{P}(5,7,8,20) \supset (40)$	$Z^2, W^4Z, WX^5, W^5XY, Y^5$	$E_8 \perp T_{2,5,5}$
86	$\boldsymbol{P}(4,5,7,9) \supset (25)$	$YZ^2, W^4Z, X^5, W^5X, WY^3$	(18)
46	$\boldsymbol{P}(5,6,22,33) \supset (66)$	$Z^2, W^{12}X, X^{11}, Y^3$	$E_8^2 \perp U$
65	$\boldsymbol{P}(3,5,11,14) \supset (33)$	XZ^2, W^{11}, WX^6, Y^3	(18)
80	$\boldsymbol{P}(4,5,13,22) \supset (44)$	Z^2, W^{11}, WX^8, XY^3	
56	$\boldsymbol{P}(5,6,8,11) \supset (30)$	YZ^2, W^6, X^5, XY^3	$E_8^2 \perp A_1 \perp U$
73	$\boldsymbol{P}(7,8,10,25) \supset (50)$	Z^2, W^6X, X^5Y, Y^5	(19)

Table 2 : Monomial transformations of the weighted projective spaces.

When Δ is symmetric, other monomial transformations exist; in the list below, the monomials in bold can be exchanged in a row.

No.	Families	The vertices of Δ	Picard lattice
16	$\boldsymbol{P}(3,6,7,8) \supset (24)$	$\boldsymbol{Z^3}, W^3YZ, W^6X, X^4, \boldsymbol{WY^3}$	$E_8 \perp (A_2)^3 \perp U$
54	$\boldsymbol{P}(3,5,6,7) \supset (21)$	$\boldsymbol{Z^3}, W^3XZ, W^7, WY^3, \boldsymbol{X^3Y}$	(16)
30	$\boldsymbol{P}(5,7,8,20) \supset (40)$	$Z^2, W^4Z, \boldsymbol{WX^5}, W^5XY, \boldsymbol{Y^5}$	$E_8 \perp T_{2,5,5}$
86	$\boldsymbol{P}(4,5,7,9) \supset (25)$	$YZ^2, W^4Z, \boldsymbol{X^5}, W^5X, \boldsymbol{WY^3}$	(18)
46	$\boldsymbol{P}(5,6,22,33) \supset (66)$	$Z^2, \boldsymbol{W^{12}X}, \boldsymbol{X^{11}}, Y^3$	$E_8^2 \perp U$
65	$\boldsymbol{P}(3,5,11,14) \supset (33)$	$XZ^2, \boldsymbol{W^{11}}, \boldsymbol{WX^6}, Y^3$	(18)
80	$\boldsymbol{P}(4,5,13,22) \supset (44)$	$Z^2, \boldsymbol{W^{11}}, \boldsymbol{WX^8}, XY^3$	
56	$\boldsymbol{P}(5,6,8,11) \supset (30)$	$YZ^2, \boldsymbol{W^6}, \boldsymbol{X^5}, \boldsymbol{XY^3}$	$E_8^2 \perp A_1 \perp U$
73	$\boldsymbol{P}(7,8,10,25) \supset (50)$	$Z^2, \boldsymbol{W^6X}, \boldsymbol{X^5Y}, \boldsymbol{Y^5}$	(19)

Table 3 : Other monomial transformations.

4. Remarks

1) Associated birational maps can be taken as monomial maps. (see Table 2)
2) There are more than one way to take Δ. (see Table 3).
For example, let us look at pairs (No. 16, No. 54) (Figure 2) and (No. 26, No. 34, No. 76) (Figure 3 and 4).

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